Q: $The\ line\ with\ equation\ ax + by = c\ is\ perpendicular\ to\ the\ line\ with\ equation\ 2x + 3y = 5$

$and\ intersects\ it\ at\ (4, -1).$

$If\ |a|\ and\ |b|\ are\ integers\ with\ greatest\ common\ factor\ 1, find\ |a| + |b| + |c|.$

A:

$The\ line\ with\ the\ equation\ 2x + 3y = 5\ has\ a\ slope\ of\ -\frac{2}{3}$

$and\ a\ perpendicular\ line\ has\ a\ slope\ of\ \frac{3}{2}.$

$Given\ that\ (4, -1) is\ a\ point\ on\ the\ line, the\ equation\ of\ the\ perpendicular\ line\ is$

$y + 1 = \frac{3}{2}(x - 4).$

$Coefficients\ a\ and\ b\ are\ integers, with\ no\ common\ factors > 1,$

$so\ the\ equivalent\ equation\ is\ 3x - 2y = 14\ and\ |a| + |b| + |c| = 19.$

Q: $A\ club\ has\ N\ members.\ The\ club\ finds\ that\ the\ number\ of\ ways\ to\ staff\ the\ 8 - member$

$executive\ committee\ is\ 42\ times\ the\ number\ of\ ways\ to\ staff\ the\ 6 - member$

$nominating\ committee.\ Find\ N.$

A: $An\ equation\ to\ represent\ this\ situation\ is$: $\frac{N!}{(N-8)!8!} = 42\frac{N!}{(N-6)!6!}$.

Multiplying both sides of the equation by 8! yields: $\frac{N!}{(N-8)!} = 42 \cdot 8 \cdot 7 \frac{N!}{(N-6)!}$

Multiplying both sides of this equation by (N − 6)! yields: $\mathrm{N}!\,(\mathrm{N}-6)(\mathrm{N}-7) = 42 \cdot 8 \cdot 7 \cdot N!.$

$Dividing\ out\ N!\ gives\ the\ quadratic\ equation\ N^2 - 13N - 2310 = 0,$

$which\ has\ a\ positive\ solution\ at\ N = 55.$

Q: $Let \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{512}. \ What\ are\ the\ last\ two\ digits\ of\ b?$

A: $Observing\ the\ first\ few\ powers\ of\ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \ldots$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} \ ; \ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^3 = \begin{bmatrix} 1 & 7 \\ 0 & 8 \end{bmatrix} \ ; \ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^4 = \begin{bmatrix} 1 & 15 \\ 0 & 16 \end{bmatrix} \ ; \ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^5 = \begin{bmatrix} 1 & 31 \\ 0 & 32 \end{bmatrix} \ ; \ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^6 = \begin{bmatrix} 1 & 63 \\ 0 & 64 \end{bmatrix} \ ; \ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^7 = \begin{bmatrix} 1 & 127 \\ 0 & 128 \end{bmatrix} \ ; \ \ldots$$

$$So, this\ question\ is\ asking\ for\ the\ last\ 2\ digits\ of\ 2^{512} - 1.$$

$$The\ following\ obeservations\ can\ be\ applied:$$

$$When\ n\ is\ a\ multiple\ of\ 4, the\ last\ digit\ of\ 2^n - 1\ is\ a\ 5.$$

$$Further, when\ n\ is\ a\ multiple\ of\ 4:$$

$$if\ \frac{n}{4}\ mod\ 5 = 1, the\ last\ 2\ digits\ of\ 2^n - 1\ are\ 15;$$

$$if\ \frac{n}{4}\ mod\ 5 = 2, the\ last\ 2\ digits\ of\ 2^n - 1\ are\ 55;$$

$$if\ \frac{n}{4}\ mod\ 5 = 3, the\ last\ 2\ digits\ of\ 2^n - 1\ are\ 95;$$

$$if\ \frac{n}{4}\ mod\ 5 = 4, the\ last\ 2\ digits\ of\ 2^n - 1\ are\ 35;$$

$$if\ \frac{n}{4}\ mod\ 5 = 0, the\ last\ 2\ digits\ of\ 2^n - 1\ are\ 75.$$

$When\ n = 512,\ \frac{512}{4} = 128\ and\ 128\ mod\ 5 = 3, so\ the\ last\ two\ digits\ of\ 2^{512} - 1\ are\ 95.$