Problem Corner Solutions

1. If the line with equation y = 2x + 5 is reflected about the line with equation y = x + 1, the reflected line has the equation y = mx + b. Find m + b.

Knowing that graphs of inverse functions are reflections about the line $y = x$, shift the line $y = 2x + 5$ and the reflected line $y = x + 1$ down one unit. This leaves the line $y = 2x + 4$ reflected about the line $y = x$. Use your favorite way of finding function inverses to determine that the reflected line has the equation $y = \frac{1}{2}x - 2$. Now shift all the equations back up one unit and the reflected line has the equation $y = \frac{1}{2}x - 1$.

The sum m + b = $\frac{1}{2} - 1 = -\frac{1}{2}$

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Or, do a substitution: $u = x + 1$

Original line: $y = 2x + 5 \rightarrow y = 2(u - 1) + 5 \rightarrow y = 2u + 3$

Line of reflection: $y = x + 1 \rightarrow y = u$

Inverse of original line: $y = \frac{1}{2}(u - 3)$

Back substitute: $y = \frac{1}{2}((x + 1) - 3) \rightarrow y = \frac{1}{2}(x - 2) \rightarrow y = \frac{1}{2}x - 1$

2. Different shades of pink, red, and white can be made by mixing whole numbers of quarts of red and/or white paint. Shades are different if the ratio of red to white paint is different. Find the number of different possible shades that can be made from at most 6 quarts of red and 5 quarts of white paint.

Let $\frac{r}{w}$ represent the ratio of quarts of red paint to quarts of white paint when neither r nor w is zero. The value of r is a whole number between 1 and 6, inclusive. The value of w is a whole number between 1 and 5, inclusive. This yields $6 \cdot 5 = 30$ ratios, each representing a shade of pink. However, some of these ratios are not simplified – we exclude the ratios where r and w have a common prime factor.

$\frac{6}{4} = \frac{3}{2}$; $\frac{6}{3} = \frac{4}{2} = \frac{2}{1}$; $\frac{6}{2} = \frac{3}{1}$; $\frac{2}{4} = \frac{1}{2}$ Excludes 5 ratios

$\frac{5}{5} = \frac{4}{4} = \frac{3}{3} = \frac{2}{2} = \frac{1}{1}$ Excludes 4 ratios

This leaves 21 different shades.

When r = 0 (no red paint), the final mixture will be white, regardless of the value of w.

When w = 0 (no white paint), the final mixture will be red, regardless of the value of r.

This creates two more shades for a total of 23 different shades of paint.

3. If $x + y = a$ and $x^2 + y^2 = b$, the expression $x^4 + y^4$ can be written as a polynomial of the form $pa^4 + qa^2b + rb^2$, where p, q and r are rational constants. Find pqr.

Given a and b as above:

$$a^4 = (x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$a^2b = (x+y)^2(x^2+y^2) = x^4 + 2x^3y + 2x^2y^2 + 2xy^3 + y^4$$

$$b^2 = (x^2+y^2)^2 = x^4 + 2x^2y^2 + y^4$$

Then, $x^4 + y^4 = pa^4 + qa^2b + rb^2 =$

$$px^4 + 4px^3y + 6px^2y^2 + 4pxy^3 + py^4 +$$

$$qx^4 + 2qx^3y + 2qx^2y^2 + 2qxy^3 + qy^4 +$$

$$rx^4 \quad + 2rx^2y^2 \quad + ry^4$$

Equating the coefficients of like terms leads to:

$$\begin{aligned} x^4&: \quad p + q + r = 1 \\ x^3y&: \quad 4p + 2q = 0 \\ x^2y^2&: \quad 6p + 2q + 2r = 0 \\ xy^3&: \quad 4p + 2q = 0 \\ y^4&: \quad p + q + r = 1 \end{aligned}$$

Solve the system to find p, q and r:

$$\begin{cases} p + q + r = 1 \\ 4p + 2q = 0 \\ 6p + 2q + 2r = 0 \end{cases}$$

$$p = -\frac{1}{2} \qquad q = 1 \qquad r = \frac{1}{2}$$

So, $pqr = -\frac{1}{4}$